TORUS KNOTS ARE RIGID STRING INSTANTONS

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We show that (p, q) torus knots satisfy the self-dual equations of rigid string instantons and we calculate the self-intersection number of the first few knots. The results are consistent with an interpretation in terms of the first Chern number.

1. Introduction

It has been argued by Ambjørn and Durhuus [1] that "regularized bosonic strings need extrinsic curvature". This paper considers some of the consequences demonstrated by Polyakov [2,3] of adding an extrinsic curvature term to the string theory action. In particular we investigate the instantons associated with this new fine structure of strings in four-dimensional euclidean space. The self-dual equations are reduced to a simple form and torus knots are shown to lead to an interesting class of solutions.

Polyakov gives a formula for calculating the number of self-intersections of the string worldsheet. This formula is applied by computer to closed knotted strings and a very simple equation relating the number of self-intersections ν to the two winding numbers of a torus knot is conjectured. Following Mazur and Nair [4], who consider rigid string instantons as a possible string theoretic way of realising the effects of QCD ϑ vacua, we show why the instanton number should be taken to be $\frac{1}{4}\nu$.

First we show how to derive the relevant curvature relations. Then we introduce the rigid string and rigid particle actions. In section 4 we show how to obtain and solve self-dual particle and string equations. In the final section we define (p, q) torus knots and derive the empirical formula for the instanton number $Q=q-p$.

2. Extrinsic geometry

The reparametrisation invariant distance between two points P and Q on a surface $\mathcal{M}^2$ parametrised by curvilinear coordinates $\xi^a=(\tau, \sigma)$ and embedded in a flat higher dimensional space is $\int_P^Q \mathrm{d}s$, where

$$\mathrm{d}s^2=\eta_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu$$
$$=\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^a}\frac{\partial X^\nu}{\partial\xi^b}\mathrm{d}\xi^a\,\mathrm{d}\xi^b\equiv g_{ab}\,\mathrm{d}\xi^a\,\mathrm{d}\xi^b\,. \tag{1.1}$$

This is the first fundamental quadratic form associated with the surface and, writing $\partial_\alpha\equiv\partial/\partial\xi^a$,

$$g_{ab}(X)\equiv\eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu \tag{1.2}$$

is the induced metric tensor.

Consider the case $\mu=1, 2, 3$; that is $\mathcal{M}^2\to\mathbb{R}^3$. There is a second fundamental quadratic form connected with such an embedding. It relates to the vertical drop associated with translations on the surface and it can be defined as

$$\mathrm{d}s\,\mathrm{d}\vartheta=-\eta_{\mu\nu}\mathrm{d}X^\mu\,\mathrm{d}N^\nu$$
$$=-\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^a}\frac{\partial N^\nu}{\partial\xi^b}\mathrm{d}\xi^a\mathrm{d}\xi^b\equiv K_{ab}\mathrm{d}\xi^a\,\mathrm{d}\xi^b\,, \tag{1.3}$$

where $N^\mu(\xi)$ is the unit vector perpendicular to the tangent plane at any point $x^\mu(\xi)$ on the surface so that the vectors $(\partial_\tau x^\mu, \partial_\sigma x^\mu, N^\mu)$ form a moving triad basis. In three dimensions

$$N^\mu=\frac{\partial_\tau X^\mu\wedge\partial_\sigma X^\mu}{|\partial_\tau X^\mu\wedge\partial_\sigma X^\mu|}\,. \tag{1.4}$$

We define

$$K_{ab}(X, N) \equiv -\eta_{\mu\nu}\,\partial_a X^\mu \partial_b N^\nu \tag{1.5}$$

as the extrinsic curvature tensor. Dividing (1.3) by (1.1) gives the normal curvature $\varkappa = \mathrm{d}\vartheta/\mathrm{d}s$ which can be regarded as a function of $\lambda = \mathrm{d}\sigma/\mathrm{d}\tau$, the slope of a line on the surface. The extreme values of $\varkappa(\lambda)$ lead to the definitions of mean curvature $\frac{1}{2}(\varkappa_{\max} + \varkappa_{\min})$ and total or gaussian curvature $\varkappa_{\max}\varkappa_{\min}$. Note that the total curvature of a cylinder or a corrugated sheet or a plane with a straight crease in it is zero!

Expressing $\partial_a\partial_b X^\mu$ in the moving triad $(\partial_\tau X^\mu, \partial_\sigma X^\mu, N^\mu)$ leads to the Gauss equations which are differential equations relating the components of g_{ab} and K_{ab} (see e.g. ref. [5]).

$$\partial_a\partial_b X^\mu = \Gamma^c{}_{ab}\,\partial_c X^\mu + K_{ab} N^\mu\,. \tag{1.6}$$

Since $\partial_a N^\mu$ lie in the tangent plane to the surface, they can be written as linear combinations of $\partial_\tau X^\mu$ and $\partial_\sigma X^\mu$. The coefficients can be expressed in terms of the components of g_{ab} and K_{ab} giving the Weingarten equations [5]

$$\partial_a N^\mu = -K_{ab} g^{bc}\,\partial_c X^\mu\,. \tag{1.7}$$

Applying the identity $\partial_a(\partial_b\partial_c X^\mu) \equiv \partial_b(\partial_a\partial_c X^\mu)$ to the Gauss equations (1.6) and using the Weingarten relations (1.7) leads to the Codazzi–Mainardi equations [5]

$$\partial_a K_{bc} - \Gamma^d{}_{ba} K_{dc} = \partial_c K_{ba} - \Gamma^d{}_{bc} K_{da}\,, \tag{1.8}$$

where $\Gamma^a{}_{bc}$ are the usual Christoffel symbols

$$\Gamma^a{}_{bc}(g) = \tfrac{1}{2} g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc})\,. \tag{1.9}$$

Using the equations of Gauss (1.6) and Weingarten (1.7) in the usual expression for the intrinsic curvature scalar associated with the surface

$$\begin{aligned} R &= g^{ab} R^c{}_{acb} \\ &= g^{ab}(\partial_c \Gamma^c{}_{ab} - \partial_b \Gamma^c{}_{ac} + \Gamma^e{}_{ab}\Gamma^c{}_{ec} - \Gamma^e{}_{ac}\Gamma^c{}_{eb}) \end{aligned} \tag{1.10}$$

gives

$$R(g) = (K^a{}_a)^2 - (K^a{}_b K^b{}_a)\,, \tag{1.11}$$

which relates intrinsic to extrinsic curvature.

All these equations can be generalised to the case of a surface embedded in four-dimensional euclidean space, $\mathcal{M}^2 \to \mathbb{R}^4$. Here the space complementary to the surface is two dimensional so we must introduce two normals at each point $N^{A\mu}(\xi)$; $A = 1, 2$; $\mu = 1, 2, 3, 4$. They are chosen to be orthogonal to the tangent space vectors

$$N^{A\mu}\partial_a X^\mu = 0 \tag{1.12}$$

and mutually orthonormal

$$N^{A\mu} N^{B\mu} = \delta^{AB}\,. \tag{1.13}$$

There are then two extrinsic curvature tensors $K^A{}_{ab}$ defined at each point in $\mathcal{M}^2$. Expressing $\partial_a\partial_b X^\mu$ in the moving tetrad basis $(\partial_\tau X^\mu, \partial_\sigma X^\mu, N^{1\mu}, N^{2\mu})$ gives the generalised Gauss equations [2]

$$\partial_a\partial_b X^\mu = \Gamma^c{}_{ab}\,\partial_c X^\mu + K^A{}_{ab} N^{A\mu} \tag{1.14}$$

and Weingarten equations [6]

$$\partial_a N^{A\mu} = -(N^{A\nu}\partial_a N^{B\nu}) N^{B\mu} - K^A{}_{ab} g^{bc}\,\partial_c X^\mu\,. \tag{1.15}$$

The relation between the curvature scalar of the riemannian manifold $\mathcal{M}^2$ and the extrinsic curvature tensor associated with its embedding is now [2]

$$R = (K^{Aa}{}_a)^2 - K^{Aa}{}_b K^{Ab}{}_a\,. \tag{1.16}$$

3. Rigidity

String theory normally begins from the Nambu–Goto action

$$S_1 = \mu \iint \mathrm{d}^2\xi \sqrt{g}\,, \tag{2.1}$$

where μ has dimensions of force $[\mathrm{ML}^{-1}]$ $(c=1)$ and is interpreted as the constant string worldsheet surface tension.

$$\begin{aligned} \sqrt{g} &\equiv \sqrt{\det g_{ab}} \\ &= [(\partial_\tau X)^2(\partial_\sigma X)^2 - (\partial_\tau X\cdot\partial_\sigma X)^2]^{1/2} \end{aligned} \tag{2.2}$$

is the area of a parallelogram with sides $\partial_\tau X$ and $\partial_\sigma X$. Therefore $\mathrm{d}A = \sqrt{g}\,\mathrm{d}\tau\,\mathrm{d}\sigma$ defines an element of area of the string worldsheet. Taking the string action to be $\propto \iint \mathrm{d}A$ implies a principle of least worldsheet area analagous to the particle action $\propto \int \mathrm{d}s$ implying a principle of shortest worldline length.

However, S_1 can also be viewed as a cosmological term in the action for the metric tensor field g_{ab} on $\mathcal{M}^2$, μ being the cosmological constant. From this point of view the Einstein term $\propto \int\int d^2\xi\sqrt{g}\,R$ should also be added. In the case of a two-dimensional manifold $\mathcal{M}^2$ the integrand is a total divergence and the integral is the Euler characteristic, $\chi(p)=2-2p$, where p is the (constant) genus of $\mathcal{M}^2$. So the Einstein term does not influence free string dynamics.

Polyakov [2] noticed that the individual terms on the right-hand side of (1.16) are not total divergencies although they are related by a total divergence which makes them equivalent under integration. This leads to the unique scale invariant generalisation of the Nambu–Goto action (2.1)

$$S=S_1+S_2$$
$$=\mu\int\int d^2\xi\sqrt{g}+\rho\int\int d^2\xi\sqrt{g}K^{Aa}{}_b K^{Ab}{}_a . \quad (2.2)$$

The new constant ρ has dimensions [ML] ($N^{A\mu}$ are unit vectors, $\mu=1, ..., D$; $A=1, ..., D-2$). ρ is interpreted as string worldsheet rigidity because it measures the opposition to extrinsic worldsheet bending. S_2 has the dynamical role of distinguishing smooth worldsheets of a given area from creased worldsheets of the same area.

S_2 can be rewritten as

$$S_2=\rho\int\int d^2\xi\,\sqrt{g}\,g^{ab}\partial_a t^{\mu\nu}\partial_b t^{\mu\nu} , \quad (2.4)$$

where

$$t^{\mu\nu}=\frac{\epsilon^{ab}}{\sqrt{g}}\partial_a X^\mu\partial_b X^\nu . \quad (2.5)$$

Alternatively, S_2 can be written as

$$S_2=\rho\int\int d^2\xi\sqrt{g}\,g^{ab}\nabla_a N^{A\mu}\nabla_b N^{A\mu} , \quad (2.6)$$

where

$$\nabla_a N^{A\mu}=\partial_a N^{A\mu}+(N^{A\nu}\partial_a N^{B\nu})N^{B\mu} \quad (2.7)$$
$$=-K^A{}_{ab}g^{bc}\partial_c X^\mu , \quad (2.8)$$

from (1.15).

By analogy with the rigid string, we can introduce rigid point particles (see e.g. ref. [7]). We write the action as

$$S=S_1+S_2$$
$$=m\int d\tau\sqrt{g}+\lambda^2\int d\tau\frac{1}{\sqrt{g}}\frac{dt^\mu}{d\tau} , \quad (2.9)$$

where

$$t^\mu=\frac{1}{\sqrt{g}}\frac{dX^\mu}{d\tau} . \quad (2.10)$$

Compare this with (2.4) and (2.5). S_1 is the usual relativistic particle action which scales as [L] while S_2 is also reparametrisation invariant and scales as [L^{-1}], a curvature. λ^2 has dimensions [ML^2], an inertia.

4. Instantons

Consider a rigid particle in two dimensions, $\mu=1$, 2. Noticing from (2.10) that $t^2=1$ we can write the action (2.9) as

$$S=\int d\tau\, m\sqrt{g}\left(t^\mu\mp\frac{\lambda\epsilon^{\mu\nu}}{\sqrt{m}\sqrt{g}}\frac{dt^\nu}{d\tau}\right)$$
$$\pm\int d\tau\, 2\lambda\sqrt{m}\,\epsilon^{\mu\nu}t^\mu\frac{dt^\nu}{d\tau} . \quad (3.1)$$

So the action is bounded by a topological part and this bound is attained for

$$t^\mu=\pm\frac{\lambda\epsilon^{\mu\nu}}{\sqrt{m}\sqrt{g}}\frac{dX^\nu}{d\tau} . \quad (3.2)$$

Solving for X^μ and picking a convenient origin gives

$$X^\mu=\pm\frac{\lambda\epsilon^{\mu\nu}}{\sqrt{m}\sqrt{g}}\frac{dX^\nu}{d\tau} , \quad (3.3)$$

which describes a circle radius $\lambda/\sqrt{m}$. This finite action solution is an instanton (anti-instanton) with parametric solution

$$X^\mu=\frac{\lambda}{\sqrt{m}}(\cos\vartheta(\tau) , \sin\vartheta(\tau)) , \quad (3.4)$$

which represents closed self-intersecting worldlines on a plane.

The topological part of the action gives the algebraic total number of times the particle worldline intersects itself.

We can apply similar arguments to the rigid string action. In particular (2.4) can be expressed as

$$S_2=\rho\int\int d^2\xi\sqrt{g}\,g^{ab}[\tfrac{1}{2}\partial_a(t^{\mu\nu}\mp{}^*t^{\mu\nu})$$
$$\times\partial_b(t^{\mu\nu}\mp{}^*t^{\mu\nu})\pm\partial_a{}^*t^{\mu\nu}\partial_b t^{\mu\nu}]\,, \tag{3.5}$$

where

$${}^*t^{\mu\nu}\equiv\tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}t^{\rho\lambda}\,. \tag{3.6}$$

S_2 is bounded by a topological part to the action and this bound is attained for

$$\partial_a(t^{\mu\nu}\mp{}^*t^{\mu\nu})=0\,. \tag{3.7}$$

We can find (see ref. [8]) instanton solutions of

$$\tau^{\mu\nu}-{}^*t^{\mu\nu}=c^{\mu\nu}\,, \tag{3.8}$$

where $c^{\mu\nu}$ are the constants from integration of (3.7).

We choose an euclidean conformal gauge in which

$$\partial_\tau X^\mu\partial_\sigma X^\mu=0 \tag{3.9}$$

and

$$(\partial_\tau X)^2=(\partial_\sigma X)^2\,. \tag{3.10}$$

Then (3.8) becomes

$$(\partial_\tau X^\mu\partial_\sigma X^\nu-\partial_\tau X^\nu\partial_\sigma X^\mu)$$
$$-\tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}(\partial_\tau X^\rho\partial_\sigma X^\lambda-\partial_\tau X^\lambda\partial_\sigma X^\rho)$$
$$=c^{\mu\nu}(\partial_\sigma X)^2\,. \tag{3.11}$$

Contracting (3.11) with $\partial_\tau X^\mu$ gives

$$\partial_\sigma X^\nu=c^{\mu\nu}\partial_\tau X^\mu\,. \tag{3.12}$$

Contracting (3.11) with $\partial_\sigma X^\mu$ gives

$$\partial_\tau X^\nu=-c^{\mu\nu}\partial_\sigma X^\mu\,. \tag{3.13}$$

Combining (3.12) and (3.13) gives

$$c^{\mu\nu}c^{\rho\mu}=-\delta^{\nu\rho}\,. \tag{3.14}$$

Also since $t^{\mu\nu}$ and ${}^*t^{\mu\nu}$ are antisymmetric, so is $c^{\mu\nu}$ from (3.8).

$$c^{\mu\nu}=-c^{\nu\mu}\,. \tag{3.15}$$

A solution to (3.14) and (3.15) is

$$c^{\mu\nu}=\begin{pmatrix}0&-\alpha&0&0\\ \alpha&0&0&0\\ 0&0&0&-\beta\\ 0&0&\beta&0\end{pmatrix}, \tag{3.16}$$

in which $\alpha^2=\beta^2=1$.

Substituting this into (3.11) we find $\beta=-\alpha$. Then (3.12) and (3.13) give

$$\partial_\tau X^1=-\alpha\partial_\sigma X^2\,,$$
$$\partial_\tau X^2=\alpha\partial_\sigma X^1\,,$$
$$\partial_\tau X^3=\alpha\partial_\sigma X^4\,,$$
$$\partial_\tau X^4=-\alpha\partial_\sigma X^3\,. \tag{3.17}$$

These are the rigid string instanton equations. Note that any X^μ satisfying (3.17) automatically satisfies the euclidean string constraint equations (3.9) and (3.10). By differentiating (3.17) we see that the string equation of motion

$$\partial_\tau\partial_\tau X^\mu+\partial_\sigma\partial_\sigma X^\mu=0 \tag{3.18}$$

is satisfied automatically.

Take $\alpha=1$ and notice that the first two relationships in (3.17) are the Cauchy–Riemann relations for an analytic function

$$F(\xi)=X^2(\xi)+iX^1(\xi)\,, \tag{3.19}$$

where $\xi=\tau+i\sigma$. Similarly the last two relationships in (3.17) are the Cauchy–Riemann relations for

$$G(\xi)=X^3(\xi)+iX^4(\xi)\,. \tag{3.20}$$

Thus we can generate an instanton solution from any two complex analytic functions F and G as

$$X^\mu=(\mathrm{Im}\,F,\mathrm{Re}\,F,\mathrm{Re}\,G,\mathrm{Im}\,G)\,. \tag{3.21}$$

As in the case of the particle in two dimensions where the topological part of the action was interpreted as giving the algebraic number of self-intersections of the euclidean particle worldline, so in the rigid string case the topological part of the action can be interpreted as the algebraic number of self-intersections of the string worldsheet. For instantons S_2 of (3.5) becomes

$$S_2=\rho\int\int d^2\xi\sqrt{g}\,g^{ab}\partial_a{}^*t^{\mu\nu}\partial_b t^{\mu\nu}\,. \tag{3.22}$$

Polyakov [2] gives the self-intersection number of the surface as

$$\nu(\mathcal{M}^2) = \frac{S_2}{2\pi\rho}$$

$$= \frac{1}{2\pi}\int\int d^2\xi \sqrt{g}\, g^{ab}\partial_a {}^*t^{\mu\nu}\partial_b t^{\mu\nu}, \tag{3.23}$$

which Mazur and Nair [4] rewrite as

$$\nu(\mathcal{M}^2) = \frac{1}{\pi}\int\int d^2\xi\, g^{cd}\epsilon^{ab}\epsilon_{AB}K^A{}_{ac}K^B{}_{bd}. \tag{3.24}$$

5. Torus knots

We are interested to find an example of a self-dual string worldsheet with a finite non-zero number of self-intersections. Taking $F \propto G$ gives $\nu=0$ so this would be a trivial example.

However, a string with a knot in it would seem to offer hope of providing an example of a worldsheet with a few undeniable self-intersections, the number increasing with the complexity of the knot.

Begin by considering the simplest possible knot, the trefoil of fig. 1a. This can be generated from

$$(u, v) \in \mathbb{C}^2: \quad u^2 + v^3 = 0. \tag{4.1}$$

Consider two orthogonal complex planes with arbitrary points u on the first and v on the second. If we consider the set of pairs (u, v) for which the relationship $u^2+v^3=0$ holds, then the intersection of this set of points with a small sphere S^3_ϵ centered at the origin of $\mathbb{C}^2$ is a trefoil knot (see e.g. ref. [9]).

As the S^3_ϵ sphere gets a little bigger so the knot gets bigger, but as S^3_ϵ gets smaller the knot contracts to a point giving us a singularity at the origin. It will be by integrating over such singularities that our self-intersection number, ν, will take on integral values.

The solution of $u^2+v^3=0$ in terms of a complex parameter z can be written as

$$u = z^3, \quad v = -z^2. \tag{4.2}$$

The trefoil can therefore be specified in terms of two analytic functions z^3 and $-z^2$.

Take the functions in (4.2) as our required functions F and G in (3.21), where $z = \tau + i\sigma$.

$$X^\mu = (\mathrm{Im}\, z^3, \mathrm{Re}\, z^3, \mathrm{Re} -z^2, \mathrm{Im} -z^2),$$

$$\therefore X^\mu = (3\tau^2\sigma - \sigma^3, \tau^3 - 3\tau\sigma^2, \sigma^2 - \tau^2, -2\tau\sigma), \tag{4.3}$$

$$\therefore \partial_\tau X^\mu = (6\tau\sigma, 3(\tau^2-\sigma^2), -2\tau, -2\sigma),$$

$$\partial_\sigma X^\mu = (3(\tau^2-\sigma^2), -6\tau\sigma, 2\sigma, -2\tau). \tag{4.4}$$

The string constraints and equation of motion are satisfied by (4.3).

Calculate $\sqrt{g}$ and $\partial_\tau X^\mu \partial_\sigma X^\nu$ and hence find $t^{\mu\nu}$ and $\partial_a t^{\mu\nu}$. Since we are considering instanton solutions for which, from (3.7),

$$\partial_a t^{\mu\nu} = \partial_a {}^*t^{\mu\nu} \tag{4.5}$$

and since we are working in the euclidean conformal gauge, the expression for the self-intersection number simplifies to

$$\nu = \frac{1}{2\pi}\int\int d^2\xi\, (\partial_a t^{\mu\nu})^2. \tag{4.6}$$

For the trefoil we find that

$$(\partial_a t^{\mu\nu})^2 = \frac{288}{[9(\tau^2+\sigma^2)+4]^2}. \tag{4.7}$$

Integrating this over all $0 \leqslant \sigma \leqslant 2\pi$ for a closed string and $-\infty \leqslant \tau \leqslant \infty$ we find

$$\nu(\text{trefoil}) = 4. \tag{4.8}$$

The construction of the trefoil based on the complex curve $u^2+v^3=0$ can be generalised to the construction of all (p, q) torus knots based on the complex curves $u^p+v^q=0$ where p and q are chosen to be relatively prime. Thus the trefoil is a (2,3) torus knot.

Following the calculation for the trefoil we take for the (2,5) torus knot

$$X^\mu = (\mathrm{Im}\, z^5, \mathrm{Re}\, z^5, \mathrm{Re} -z^2, \mathrm{Im} -z^2). \tag{4.9}$$

This gives

$$\nu(2,5)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{2\pi} d\tau\, d\sigma \frac{7200(\tau^2+\sigma^2)^2}{[25(\tau^2+\sigma^2)^3+4]^2} \tag{4.10}$$

$$= \int_0^{2\pi} dr \frac{7200\, r^5}{(25r^6+4)^2} \tag{4.11}$$

$$\therefore \nu(2,5) = 12. \tag{4.12}$$

Similarly

$$\nu(2,7)$$

$$= \frac{1}{2\pi}\int\int d\tau\, d\sigma \frac{39200(\tau^2+\sigma^2)^4}{[49(\tau^2+\sigma^2)^5+4]^2}, \tag{4.13}$$

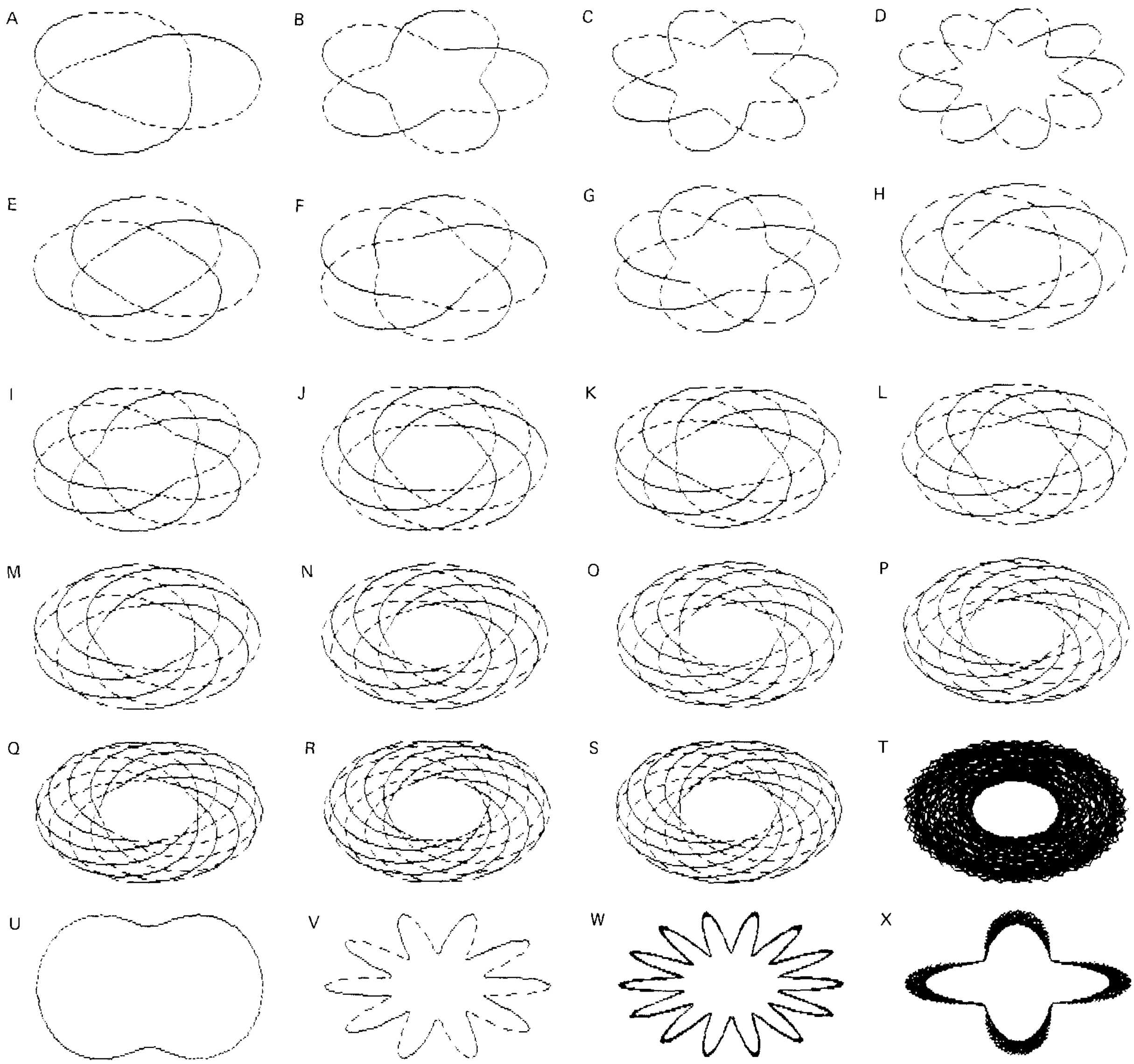

Fig. 1. Knots and links. (A) (2, 3) knot, (B) (2, 5) knot, (C) (2, 7) knot, (D) (2, 9) knot, (E) (3, 4) knot, (F) (3, 5) knot, (G) (3, 7) knot, (H) (4, 5) knot, (I) (4, 7) knot, (J) (5,6) knot, (K) (5, 7) knot, (L) (5, 8) knot, (M) (6, 7) knot, (N) (7, 8) knot, (O) (7, 9) knot, (P) (8, 9) knot, (Q) (8, 11) knot, (R) (9, 10) knot, (S) (9, 11) knot, (T) (503, 634) knot, (U) (2, 4) link, (V) (3, 30) link, (W) (7, 98) link, (X) (51, 204) link.

$$\therefore\ \nu(2,7)=20\,. \tag{4.14}$$

Results for some other knots, computed using REDUCE, are shown in table 1.

Looking at these results, the general form for $\nu(p, q)$ would seem to be given by the empirical formula

$$\nu(p,q)=\int_0^{2\pi}\mathrm{d}r\,\frac{8(q-p)^2q^2p^2r^{2(q-p-1)+1}}{(q^2r^{2(q-p)}+p^2)^2}\,. \tag{4.15}$$

So we find

Table 1
Self-intersection number ν for a (p, q) torus knot rigid string instanton

p	q	$\int dr$ integrand	$\nu/4$
2	3	$288r/(9r^2+4)^2$	1
2	5	$7200r^5/(25r^6+4)^2$	3
2	7	$39200r^9/(49r^{10}+4)^2$	5
2	9	$127008r^{19}/(81r^{14}+4)^2$	7
3	4	$1152r/(16r^2+9)^2$	1
3	5	$7200r^3/(25r^4+9)^2$	2
3	7	$56448r^7/(49r^8+9)^2$	4
4	5	$3200r/(25r^2+16)^2$	1
4	7	$56448r^5/(49r^6+16)^2$	3
5	6	$7200r/(36r^2+25)^2$	1
5	7	$39200r^3/(49r^4+25)^2$	2
5	8	$115200r^5/(64r^6+25)^2$	3
6	7	$14112r/(49r^2+36)^2$	1
7	8	$25088r/(64r^2+49)^2$	1
7	9	$127008r^3/(81r^4+49)^2$	2
8	9	$41472r/(81r^2+64)^2$	1
8	11	$557568r^5/(121r^6+64)^2$	3
9	10	$64800r/(100r^2+81)^2$	1
9	11	$313632r^3/(121r^4+81)^2$	2

$$\nu(p,q)=4(q-p)\,, \tag{4.16}$$

which represents an infinite hierarchy of knotted instantons.

Relation (4.16) (although not (4.15)) also holds true for the case of links which occur when p and q are not relatively prime.

Notice that that ν always turns out to be a multiple of 4. Mazur and Nair [4] argue that $\nu=4c_1$ where c_1 is the first Chern number and $c_1\in\mathbb{Z}$.

In the case of a two-dimensional manifold embedded in four dimensions we have a special situation. The curvature 2-form defined on $\mathcal{M}^2$ integrated over the two dimensions of $\mathcal{M}^2$ gives an integer, the Euler characteristic. Also, since the co-dimension is two, the curvature associated with the embedding connection 2-form integrated over $\mathcal{M}^2$ gives another integer, the first Chern number.

Write the covariant derivative of the normal vectors as

$$\nabla_a N^{A\mu}\equiv\partial_a N^{A\mu}+A^{AB}{}_a N^{B\mu}=-K^{Ab}{}_a\,\partial_b X^\mu \tag{4.17}$$

and the covariant derivative of the tangent vectors

$$t^\mu_a\equiv\partial_a X^\mu \tag{4.18}$$

as

$$\mathrm{D}_a t^\mu_b\equiv\partial_a t^\mu_b-\Gamma^c{}_{ab}t^\mu_c$$
$$=K^A{}_{ab}N^{A\mu} \tag{4.19}$$

from (1.14).

We see from (2.7) that

$$A^{AB}{}_a=N^{A\mu}\partial_a N^{B\mu}\,. \tag{4.20}$$

From this connection we can derive the field strength tensor

$$F^{AB}{}_{ab}=\mathrm{D}_a N^{A\mu}\mathrm{D}_b N^{B\mu}-\mathrm{D}_a N^{B\mu}\mathrm{D}_b N^{A\mu}\,. \tag{4.21}$$

The first Chern number is then defined as [4]

$$c_1\equiv\frac{1}{2\pi}\int_{\mathcal{M}^2}\mathrm{tr}\,F=\frac{1}{8\pi}\int \mathrm{d}^2\xi\,\epsilon^{AB}\epsilon^{ab}F^{AB}{}_{ab}\,. \tag{4.22}$$

Using (4.17), (4.21) and (3.24) gives

$$c_1=\tfrac{1}{4}\nu \tag{4.23}$$

which leads to the suggestion [4] that we use exp $(\mathrm{i}\theta\frac{1}{4}\nu)$ to represent the effect of ϑ vacua.

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