Self-dual lumps and octonions

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We consider the theory of self-dual three-dimensionally extended objects immersed in eight dimensions and find a large class of classical solutions.

1. (*d*; *D*)-brane dynamics

The action of a $(d;\ D)$-brane is given by the Einstein–Nambu–Goto volume integral

$$S \equiv \int L\,\mathrm{d}^d\xi = \mathfrak{M}\int \sqrt{g}\,\mathrm{d}^d\xi\,, \qquad (1.1)$$

where the induced metric g_{ab} of the immersed manifold $\mathscr{M}^d$ is

$$g_{ab}(\xi) \equiv \frac{\partial X^\mu(\xi)}{\partial \xi^a}\frac{\partial X^\nu(\xi)}{\partial \xi^b}\,G_{\mu\nu}(X)$$
$$\equiv X^\mu{}_{,a}X_{\mu,b}\,, \qquad (1.2)$$

with $\mu,\nu = 0,\ 1,\ 2,\ \ldots,\ D-1$ and $a,\ b = 0,\ 1,\ 2,\ \ldots,\ d-1$ (unless D or d is odd, in which case we count from 1 to D or 1 to d respectively). The fundamental constant of the theory, $\mathfrak{M}$, has the dimension of an extended-mass density $[ML^{1-d}]$. The speed of light is defined to be unity. $G_{\mu\nu}$ is the metric on $\mathscr{M}^D$ where $\mathscr{M}^d \hookrightarrow \mathscr{M}^D$.

Variation of this action gives the extended geodesic equation

$$g^{ab}(\partial_a X^\mu{}_{,b} - \Gamma^c_{ab}X^\mu{}_{,c} + \Gamma^\mu_{\nu\lambda}X^\nu{}_{,a}X^\lambda{}_{,b}) = 0\,. \qquad (1.3)$$

The $(d;\ D)$-brane is taken to be closed so that no boundary conditions need to be considered. The background space is taken to be euclidean so that $G_{\mu\nu} = \mathrm{Diag}(1,1,\ \ldots,\ 1)$, and thus $\mathscr{M}^d \hookrightarrow \mathbb{R}^D$. Introducing the conjugate momentum

$$P^{a\mu} \equiv \frac{\delta L}{\delta X_{\mu,a}} = \mathfrak{M}\sqrt{g}\,g^{ab}X^\mu{}_{,b}\,, \qquad (1.4)$$

the Euler–Lagrange equation of motion (1.3) reduces to

$$\partial_a P^{a\mu} = 0\,. \qquad (1.5)$$

The appropriate identities can be written

$$P^a{}_\mu X^\mu{}_{,b} = \mathfrak{M}\sqrt{g}\,\delta^a{}_b = L\delta^a{}_b\,, \qquad (1.6)$$

which corresponds to the vanishing of the hamiltonian, and

$$P^a{}_\mu P^{b\mu} = \mathfrak{M}^2 g g^{ab}\,, \qquad (1.7)$$

which is the extended object generalization of $p^2 = m^2$ for relativistic particles.

2. Self-dual (4; 8)-brane dynamics

Biran, Floratos and Savvidy [1] wrote down an expression for the conjugate momentum of a self-(doubly)-dual (3; 3)-brane. Grabowski and Tze [2] have written down an analogous expression for a self-(doubly)-dual (4; 8)-brane:

$$\tilde{P}^a{}_\mu \equiv \mathfrak{M}\frac{1}{3!}\,e^{abcd}T_{\mu\nu\rho\sigma}X^\nu{}_{,b}X^\rho{}_{,c}X^\sigma{}_{,d}\,, \qquad (2.1)$$

where $a,\ b,\ \ldots = 0,\ 1,\ 2,\ 3$ and $\mu,\ \nu,\ \ldots = 0,\ 1,\ \ldots,\ 7$. The four-dimensional permutation symbol is fixed by setting

$$e^{0123} \equiv 1\,. \qquad (2.2)$$

The completely antisymmetric tensor, $T_{\mu\nu\rho\sigma}$, was discussed by Corrigan, Devchand, Fairlie and Nuyts in

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ref. [3]. The self-dual version (i.e. formally $T = *T$, $*$ symbolizing the eight-dimensional Hodge dual) can be defined in terms of the structure constants of octonion multiplication, C_{ijk} $(i, j, k = 1, 2, ..., 7)$, and H_{ijkl}, the seven-dimensional dual of C_{ijk} (i.e. $H = *C$);

$$T_{ijkl} = H_{ijkl} = \frac{1}{3!} \varepsilon_{ijkl}{}^{mnp} C_{mnp} ,$$

$$T_{0ijk} \equiv T_{ij0k} = C_{ijk} ,$$

$$T_{i0jk} \equiv T_{ijk0} = -C_{ijk} . \tag{2.3}$$

$T_{\mu\nu\rho\sigma}$ is invariant under SO(8) transformations and supplies a measure of the lack of associativity of three octonions via the associator

$$\tfrac{1}{2}[\mathrm{i}_\mu, \mathrm{i}_\nu, \mathrm{i}_\rho] \equiv \tfrac{1}{2}[(\mathrm{i}_\mu \mathrm{i}_\nu)\mathrm{i}_\rho - \mathrm{i}_\mu(\mathrm{i}_\nu \mathrm{i}_\rho)]$$

$$= T_{\mu\nu\rho}{}^{\sigma} \mathrm{i}_\sigma . \tag{2.4}$$

C_{ijk} is invariant under G_2 transformations and supplies a measure of the lack of commutativity of two octonions via the commutator (the scalar part of which vanishes on the right hand side)

$$\tfrac{1}{2}[\mathrm{i}_\mu, \mathrm{i}_\nu] \equiv \tfrac{1}{2}(\mathrm{i}_\mu \mathrm{i}_\nu - \mathrm{i}_\nu \mathrm{i}_\mu) = C_{ij}{}^{k} \mathrm{i}_k , \tag{2.5}$$

where $\mathrm{i}_\mu = (\mathrm{i}_0, \mathrm{i}_i)$ are, respectively, the scalar identity element and the seven imaginary units of the octonions. (H_{ijkl} is invariant under SO(7) transformations.)
The self-duality equation is

$$P^{a\mu} = \tilde{P}^{a\mu} , \tag{2.6}$$

which relates the conjugate momentum $P^{a\mu}$ of (1.4) with the world-four-volume tensor density $\tilde{P}^{a\mu}$ of (2.1). Thus (2.6) can be rewritten as

$$X^{\mu}{}_{,a} = \frac{1}{3!} \varepsilon_a{}^{bcd} T^{\mu}{}_{\nu\rho\sigma} X^{\nu}{}_{,b} X^{\rho}{}_{,c} X^{\sigma}{}_{,d} , \tag{2.7}$$

in which ϵ^{abcd} is the four-dimensional permutation tensor defined from (2.2) by

$$\varepsilon^{abcd} \equiv \frac{1}{\sqrt{g}} e^{abcd} . \tag{2.8}$$

Although an attempt has been made to properly distinguish between covariant and contravariant tensor indices in $\mathscr{M}^D$, one should note that this discussion takes $\mathscr{M}^D \cong \mathbb{R}^D$ and that therefore $\sqrt{G} = 1$, so the tensorial nature of $T_{\mu\nu\rho\sigma}$ is greatly simplified.

Contracting (2.7) with $X_\mu{}^{,a}$ gives

$$\sqrt{g} = T_{\mu\nu\rho\sigma} X^{\mu}{}_{,0} X^{\nu}{}_{,1} X^{\rho}{}_{,2} X^{\sigma}{}_{,3} , \tag{2.9}$$

which is the condition necessary for (2.1) to satisfy constraint (1.6).

3. A class of solutions

The presence of the octonion associator alerts one to the existence of an octonionic algebraic structure on $\mathbb{R}^8$. Considering then $\mathscr{M}^d \hookrightarrow \mathbb{O}$, we consider X to be an octonionic function mapping $\mathbb{C}^2$ into the field $\mathbb{O}$:

$$X: \mathbb{C} \times \mathbb{C} \to \mathbb{O} , \quad \xi^a \mapsto X^\mu(\xi) .$$

Consider X to be formed from four complex analytic functions P, Q, R and S, in a way consistent with $X \in \mathbb{O}$. Let the first two be functions of the complex variable $(\xi^0 + \xi^1 \mathrm{i})$ and let the second two be functions of the complex variable $(\xi^2 + \xi^3 \mathrm{i})$. Also identify i with i_1, the first imaginary octonion unit. Then write (using $p_1 \equiv \mathrm{Re}\, P$, $p_2 \equiv \mathrm{Im}\, P$, etc.)

$$\begin{aligned} X(\xi) &= P(\xi^0 + \xi^1 \mathrm{i}_1)\mathrm{i}_0 + Q(\xi^0 + \xi^1 \mathrm{i}_1)\mathrm{i}_2 \\ &\quad + R(\xi^2 + \xi^3 \mathrm{i}_1)\mathrm{i}_4 + S(\xi^2 + \xi^3 \mathrm{i}_1)\mathrm{i}_6 \\ &= p_1(\xi^0, \xi^1)\mathrm{i}_0 + p_2(\xi^0, \xi^1)\mathrm{i}_1 \\ &\quad + q_1(\xi^0, \xi^1)\mathrm{i}_2 + q_2(\xi^0, \xi^1)\mathrm{i}_1 \mathrm{i}_2 \\ &\quad + r_1(\xi^2, \xi^3)\mathrm{i}_4 + r_2(\xi^2, \xi^3)\mathrm{i}_1 \mathrm{i}_4 \\ &\quad + s_1(\xi^2, \xi^3)\mathrm{i}_6 + s_2(\xi^2, \xi^3)\mathrm{i}_1 \mathrm{i}_6 . \end{aligned} \tag{3.1}$$

We now choose a basis in which $\mathrm{i}_1 \mathrm{i}_2 = \mathrm{i}_3$, $\mathrm{i}_1 \mathrm{i}_4 = \mathrm{i}_5$ and $\mathrm{i}_1 \mathrm{i}_6 = \mathrm{i}_7$. That is, we choose a particular set of the completely anti-symmetric structure constants, C_{ijk}, which are defined by

$$\mathrm{i}_i \mathrm{i}_j = -\delta_{ij} \mathrm{i}_0 + C_{ij}{}^{k} \mathrm{i}_k . \tag{3.2}$$

Of the 240 admissible "clockwise" bases and the 240 admissible "anti-clockwise" bases only two clockwise and two anti-clockwise contain all three triples 123, 145 and 167. They are

$$\left|\begin{matrix} 123 \\ 263 \\ 347 \\ 451 \\ 572 \\ 635 \\ 716 \end{matrix}\right| \quad \text{and} \quad \left|\begin{matrix} 123 \\ 246 \\ 365 \\ 437 \\ 514 \\ 671 \\ 752 \end{matrix}\right| ,$$

which we term "clockwise";

$$\begin{vmatrix} 123 \\ 265 \\ 357 \\ 436 \\ 514 \\ 671 \\ 742 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 123 \\ 247 \\ 375 \\ 451 \\ 562 \\ 634 \\ 716 \end{vmatrix},$$

which we term "anti-clockwise". For the following discussion we pick

$$\begin{vmatrix} 123 \\ 246 \\ 365 \\ 437 \\ 514 \\ 671 \\ 752 \end{vmatrix}$$

as the basis of octonion multiplication. The octonionic function of the four lump parameters ξ^a, $X(\xi)$, can then be written

$$\begin{aligned} X(\xi) = & p_1(\xi^0, \xi^1) i_0 + p_2(\xi^0, \xi^1) i_1 \\ & + q_1(\xi^0, \xi^1) i_2 + q_2(\xi^0, \xi^1) i_3 \\ & + r_1(\xi^2, \xi^3) i_4 + r_2(\xi^2, \xi^3) i_5 \\ & + s_1(\xi^2, \xi^3) i_6 + s_2(\xi^2, \xi^3) i_7 \end{aligned} \tag{3.3}$$

or in component form,

$$X^\mu = (p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2) . \tag{3.4}$$

The requirement that the complex functions P, Q, R and S be analytic is expressed by four pairs of Cauchy–Riemann equations, which can be regarded as integrability conditions for these self-dual lumps.

That immersions of the form (3.4) satisfy the equation of motion (1.5), the constraint equations (1.6), (1.7) and the self-duality equation (2.6) can be demonstrated by a calculation (which was performed by analytic computation). First calculate $X^\mu{}_{,a}$, incorporating the Cauchy–Riemann conditions. Then calculate g_{ab} and hence obtain

$$\begin{aligned} \sqrt{g} = & \left[\left(\frac{\partial p_1}{\partial \xi^1}\right)^2 + \left(\frac{\partial p_2}{\partial \xi^1}\right)^2 + \left(\frac{\partial q_1}{\partial \xi^1}\right)^2 + \left(\frac{\partial q_2}{\partial \xi^1}\right)^2\right] \\ & \times \left[\left(\frac{\partial r_1}{\partial \xi^3}\right)^2 + \left(\frac{\partial r_2}{\partial \xi^3}\right)^2 + \left(\frac{\partial s_1}{\partial \xi^3}\right)^2 + \left(\frac{\partial s_2}{\partial \xi^3}\right)^2\right] . \end{aligned} \tag{3.5}$$

Use g_{ab} to find $P^{a\mu}$ by (1.4). Differentiate $P^{a\mu}$ to verify (1.5). Also verify (1.6) and (1.7).

It remains to prove (2.6). Given the definition of $T_{\mu\nu\rho\sigma}$ in (2.3) and the choice of basis

$$\begin{vmatrix} 123 \\ 246 \\ 365 \\ 437 \\ 514 \\ 671 \\ 752 \end{vmatrix},$$

one can calculate $\tilde{P}^{a\mu}$ from the definition (2.1), and hence demonstrate that (2.7) is true for any X^μ of the form (3.4). In passing, eq. (2.9) can be verified. That this is a non-trivial relationship can be seen from the implication that $g \equiv \mathrm{Det}\, g_{ab}$ must of necessity be a perfect square. This requirement places tight constraints on the allowed forms of $X^\mu{}_{,a}$ and $G_{\mu\nu}$.

4. Conclusion

We have analysed the theory of self-dual (4; 8)-branes and found a large class of classical solutions based on four independent arbitrary analytic functions of complexified lump coordinates.

One possible avenue of research is to choose particular sets of analytic functions, like those chosen in ref. [4], in order to obtain topological quantum numbers to be identified with those ascribed to the known fundamental particles. Major issues which have to be addressed include the move from euclidean to Minkowski signature and the description of fermions by taking the Dirac square root of (1.7) or by generalizing to super p-branes [5].

Since the self-dual sector admits such a large class of solutions, it might prove worthwhile to rewrite the theory in a way which more intimately incorporates the self-duality restriction. Incorporating (2.9) into (1.1) gives

$$S=\mathfrak{M}\int T_{\mu\nu\rho\sigma}X^{\mu}{}_{,0}X^{\nu}{}_{,1}X^{\rho}{}_{,2}X^{\sigma}{}_{,3}\,\mathrm{d}^4\xi$$

$$\therefore S=\frac{\mathfrak{M}}{4!}\int T_{\mu\nu\rho\sigma}\,\mathrm{d}X^{\mu}\,\mathrm{d}X^{\nu}\,\mathrm{d}X^{\rho}\,\mathrm{d}X^{\sigma}\,, \qquad (4.1)$$

in which X is an octonionic function of four variables. An action of this form has the great advantage of freeing the theory from an undesirable square root.

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